④ $x^2p^2+3xyp+2y^2=0$.

Solution:

$$x^2p^2+3xyp+2y^2=0 \rightarrow ①$$

$a=x^2$, $b=3xy$, $c=2y^2$.

$$p=\frac{-3xy\pm\sqrt{9x^2y^2-4x^2(2y^2)}}{2x^2}$$

$$=\frac{-3xy\pm\sqrt{9x^2y^2-8x^2y^2}}{2x^2}$$

$$=\frac{-3xy\pm xy}{2x^2}$$

$$=\frac{-3xy+xy}{2x^2},\ \frac{-3xy-xy}{2x^2}.$$

$$=\frac{-2xy}{2x^2},\ \frac{-4xy}{2x^2}=-\frac{y}{x},\ \frac{-2y}{x}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y}{x};\quad \frac{dy}{dx}=-\frac{2y}{x}.$$

$$\Rightarrow \frac{dy}{y}=-\frac{dx}{x};\quad \frac{dy}{y}=-\frac{2dx}{x}.$$

Integrating, we get

$\log y=-\log x+\log c_1$; $\log y=-2\log x+\log c_2$

$\Rightarrow \log xy=\log c_1$, $\log x^2y=\log c_2$.

$\Rightarrow xy-c_1=0$, $x^2y-c_2=0$.

$\therefore$ General solution is

$$\boxed{(xy-c_1)(x^2y-c_2)=0}$$

⑤ Solve $p^2-2py-3y^2=0$.

Solution: $p^2-2py-3y^2=0$.

$$\Rightarrow (p-3y)(p+y)=0.$$

$$\Rightarrow p=3y,\ -y.$$

$$\Rightarrow \frac{dy}{dx}=3y,\quad \frac{dy}{dx}=-y.$$

$$\Rightarrow \frac{dy}{y}=3dx;\quad \frac{dy}{y}=-dx.$$

Integrating we get

$\log y=3x+c_1$; $\log y=-x+c_2$.

General solution is

$$\boxed{(\log y-3x-c_1)(\log y+x+c_2)=0}.$$

⑥ $p^2+2yp\cot x=y^2$.

Solution: $p^2+2yp\cot x-y^2=0 \rightarrow ①$

$a=1$, $b=2y\cot x$, $c=-y^2$.

$$p=\frac{-2y\cot x\pm\sqrt{4y^2\cot^2x+4y^2}}{2}$$

$$=\frac{-2y\cot x\pm\sqrt{4y^2(\cot^2x+1)}}{2}$$

$$=-y\cot x\pm y\operatorname{cosec}x.$$

$$\Rightarrow p=-y\cot x+y\operatorname{cosec}x$$

$$\Rightarrow p=-y(\cot x+\operatorname{cosec}x)\rightarrow ②$$

$$\& \ p=-y(\cot x-\operatorname{cosec}x)\rightarrow ③$$

$$② \Rightarrow \frac{dy}{dx}=-y(\cot x+\operatorname{cosec}x)$$

$$\Rightarrow \frac{dy}{y}=(-\cot x-\operatorname{cosec}x)$$

Integrating we get

$$\log y=-\int\cot x\,dx+\int-\operatorname{cosec}x\,dx.$$

$$=-\log\sin x+\log(\operatorname{cosec}x+\cot x)+\log c_1.$$

$$\Rightarrow \log y+\log\sin x-\log(\operatorname{cosec}x+\cot x)=\log c_1.$$

$$\Rightarrow \log\frac{y\sin x}{\operatorname{cosec}x+\cot x}=\log c_1.$$

$$\Rightarrow \frac{y\sin x}{\operatorname{cosec}x+\cot x}=c_1 \rightarrow ④$$

(3) $\Rightarrow p = -y(\cot x - \operatorname{cosec} x)$

$\Rightarrow \dfrac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$

$\Rightarrow \dfrac{dy}{y} = (-\cot x + \operatorname{cosec} x)\,dx$

Integrating we get

$\log y = -\log \sin x - \log(\operatorname{cosec} x + \cot x) + \log c_2$

$\Rightarrow \boxed{y \sin x (\operatorname{cosec} x + \cot x) = c_2}$ $\rightarrow$ (5)

∴ General solution is

$$\boxed{\left(\frac{y \sin x}{\operatorname{cosec} x + \cot x} - c_2\right)\left[\frac{y \sin x}{(\operatorname{cosec} x + \cot x)} - c_2\right] = 0}$$

(7) $y\left(\dfrac{dy}{dx}\right)^2 + (x-y)\dfrac{dy}{dx} - x = 0.$

Solution:

$y\left(\dfrac{dy}{dx}\right)^2 + (x-y)\dfrac{dy}{dx} - x = 0.$

$\dfrac{dy}{dx} = p$

$\Rightarrow yp^2 + (x-y)p - x = 0.$

$a = y,\ b = x-y,\ c = -x.$

$$p = \frac{-(x-y) \pm \sqrt{(x-y)^2 + 4xy}}{2y}$$

$$= \frac{-x+y \pm \sqrt{x^2+2xy+y^2}}{2y}$$

$$= \frac{-x+y \pm (x+y)}{2y}$$

$$= \frac{-x+y+x+y}{2y},\ \frac{-x+y-x-y}{2y}.$$

$$= 1,\ \frac{-x}{y}.$$

$\dfrac{dy}{dx} = 1$; $\dfrac{dy}{dx} = -x/y.$

$\Rightarrow dy = dx$; $y\,dy = -x\,dx.$

Integrating we get

$y = x + c_1$, $\dfrac{y^2}{2} = -\dfrac{x^2}{2} + c_2$;

∴ General solution is

$$\boxed{(y-x-c_1)(y^2+x^2-2c_2) = 0}$$

HW (8) $xyp^2 + (x+y)p + 1 = 0.$

Ans. $(xp+1)(yp+1) = 0$

$\Rightarrow (y + \log x - c_1)\left(\dfrac{y^2}{2} + x - c_2\right) = 0.$

(9) $xp^2 - (2x+3y)p + 6y = 0.$

Ans. $p - 2 = 0$, $p - \dfrac{3y}{x} = 0.$

$\Rightarrow (y - 2x - c_1)(y/x^3 - c_2) = 0.$

(10) $yp^2 - (x - y^2)p - xy = 0,$

Ans. $p - \dfrac{x}{y} = 0$; $p + y = 0$

$\Rightarrow (y^2 - x^2 - c_1)(y - c_2 e^{-x}) = 0.$

(11) $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$.

Solution:

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - y/x. \longrightarrow (1)$$

But $p = \frac{dy}{dx} \Rightarrow \frac{1}{p} = \frac{dx}{dy}$.

$\therefore (1) \Rightarrow p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$.

$\Rightarrow p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$.

$\Rightarrow \left(p + \frac{y}{x}\right)\left(p - \frac{x}{y}\right) = 0$.

$\Rightarrow p + \frac{y}{x} = 0$; $p - \frac{x}{y}$ = 0.

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = 0$; $\frac{dy}{dx} - \frac{x}{y} = 0$.

$\Rightarrow x\,dy + y\,dx = 0$; $y\,dy - x\,dx = 0$.

$\Rightarrow d(xy) = 0$; $y\,dy - x\,dx = 0$.

Integrating, we get $y^2 - x^2 = c_2$.

$xy = c_1$;

$\therefore$ General solution is

$$\boxed{(xy - c_1)(y^2 - x^2 - c_2) = 0.}$$

HW (12) $p(p+y) = x(x+y)$

Ans. $(2y - x^2 + c_1)(y + x + c_2 e^{-x} - 1) = 0$.

(13) $xy\left(\frac{dy}{dx}\right)^2 - (x^2+y^2)\frac{dy}{dx} + xy = 0$.

Ans. $(y - c_1 x)(y^2 - x^2 - c_2) = 0$,

(14) Solve

$$p^2 + \left(x + y - \frac{2y}{x}\right)p + \left(xy + \frac{y^2}{x^2} - y - \frac{y^2}{x}\right) = 0. \longrightarrow (1)$$

Solution:

Solving (1), we get

$p = \frac{y}{x} - y$, $p = y/x - x$.

$\Rightarrow \frac{dy}{y} = \left(\frac{1}{x} - 1\right)dx \longrightarrow (2)$; $\frac{dy}{dx} - y/x = -x. \longrightarrow (3)$

(2) $\Rightarrow \frac{dy}{y} = \left(\frac{1}{x} - 1\right) dx$.

Integrating, we get

$\log y = \log x - x + \log c$.

$\Rightarrow \log \frac{y}{x} = -x + \log c$.

$\Rightarrow \log \frac{y}{cx} = -x$.

$\Rightarrow \frac{y}{cx} = e^{-x} \Rightarrow \boxed{y = cxe^{-x}}$

(3) $\Rightarrow \frac{dy}{dx} - \frac{1}{x}\cdot y = -x$.

This is linear eqn $\left[\frac{dy}{dx} + Py = Q\right]$

Here $P = -\frac{1}{x}$, $Q = -x$.

$IF = e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = x^{-1}$.

solution is

$y(IF) = \int Q(IF)\,dx + c$,

$\Rightarrow \frac{y}{x} = \int -x \cdot \frac{1}{x}\,dx + c$.

$= \int -dx + c$.

$\Rightarrow y/x = -x + c$

$\Rightarrow \boxed{y = -x^2 + cx}$

$\therefore$ General solution is

$$\boxed{(y - cxe^{-x})(y + x^2 - cx) = 0.}$$

Given equation is

$f(x, y, p) = 0 \longrightarrow (1)$

(1) is of I order and higher degree.

Determine y from (1).

We get $y = F(x, p) \rightarrow (2)$

Differentiate (2) w.r.t. x, we get

$\frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$.

But $\frac{dy}{dx} = p$.

$\Rightarrow p = \phi\left(x, p, \frac{dp}{dx}\right) \rightarrow (3)$

(3) is I order I degree. Differential equation in two variables p & x.

Solving (3), we get

$\psi(x, p, c) = 0 \rightarrow (4)$

Eliminating p from (1) or (4), and substituting in the other equation, we get general solution.

Problem (1)

(i) Solve $y - p^4x^2 - 2px = 0$.

Solution:

Given equation is

$y - p^4x^2 - 2px = 0 \rightarrow (1)$

$\Rightarrow y = p^4x^2 + 2px \longrightarrow (2)$.

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = p^4 . 2x + x^2 . 4p^3 \frac{dp}{dx} + 2\left[p . 1 + x\frac{dp}{dx}\right]$.

$\Rightarrow p = 2p^4x + 4x^2p^3\frac{dp}{dx} + 2p + 2x\frac{dp}{dx}$.

$\Rightarrow 2x\frac{dp}{dx} + 4x^2p^3\frac{dp}{dx} + 2xp^4 + 2p - p = 0$.

$\Rightarrow 2x\frac{dp}{dx}(2xp^3+1) = -p - 2p^4x$.

$= -p(1+2xp^3)$

$\Rightarrow$

$\Rightarrow 2x\frac{dp}{dx}(2xp^3+1) = -p(2xp^3+1)$.

$\Rightarrow 2x\frac{dp}{dx} = -p. \longrightarrow (3)$

(3) is I order I degree D.E.

Using variable separable method, we get

$2\frac{dp}{p} = -\frac{dx}{x}$.

Integrating, we get

$2\log p = -\log x + \log c$.

$\Rightarrow \log p^2 + \log x = \log c$.

$\Rightarrow \boxed{p^2x = c} \longrightarrow (4)$.

(4) $\Rightarrow p^2 = c/x \Rightarrow p = \pm\sqrt{c/x}$.

Substituting in (1), we get

$y - \frac{c^2}{x^2} . x^2 - 2\sqrt{c/x} . x = 0$.

$\Rightarrow \boxed{y - c^2 - 2\sqrt{cx} = 0}$.

(1) (ii) Solve

$y - 2px + p^4x^2 = 0$.

(2) (i) Solve $y = x + a\tan^{-1}p$.

Solution:

Given $y = x + a\tan^{-1}p \longrightarrow (1)$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = 1 + a\cdot\frac{1}{1+p^2}\cdot\frac{dp}{dx}.$$

$$\Rightarrow p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}.$$

$$\Rightarrow \frac{a}{1+p^2}\frac{dp}{dx} = p-1 \longrightarrow (2).$$

(2) is I order I degree D.E.

By variable separable method

$$(2) \Rightarrow \frac{a}{1+p^2}\frac{dp}{p-1} = dx.$$

$$\Rightarrow dx = \frac{a}{(p-1)(1+p^2)}dp.$$

Integrating, we get

$$\int dx = \int \frac{a}{(p-1)(1+p^2)}dp \longrightarrow (3).$$

Resolving into partial fractions, we get

$$\frac{1}{(p-1)(1+p^2)} = \frac{A}{p-1} + \frac{Bp+C}{1+p^2} \longrightarrow (4)$$

$$1 = A(p^2+1) + (Bp+C)(p-1).$$

$p = 1 \Rightarrow 2A = 1 \Rightarrow \boxed{A = 1/2.}$

Coefficient of $p^2 \Rightarrow A + B = 0$

$\Rightarrow \boxed{B = -1/2}$

$p = 0 \Rightarrow A - C = 0 \Rightarrow \boxed{C = -1/2}$

$$\therefore (4) \Rightarrow \frac{1}{(p-1)(1+p^2)} = \frac{1}{2}\cdot\frac{1}{p-1} - \frac{1}{2}\cdot\frac{p+1}{p^2+1}$$

$\therefore (3) \Rightarrow$

$$x = a\left[\frac{1}{2}\int\frac{1}{p-1}dp - \frac{1}{2}\int\frac{p+1}{p^2+1}dp\right].$$

$$= a\left[\frac{1}{2}\int\frac{dp}{p-1} - \frac{1}{4}\int\frac{2p}{p^2+1}dp - \frac{1}{2}\int\frac{dp}{p^2+1}\right]$$

$$= a\left[\frac{1}{2}\log(p-1) - \frac{1}{4}\log(p^2+1) - \frac{1}{2}\tan^{-1}p\right] + c,$$

$\therefore$ General solution is

$$x = \frac{a}{2}\left[\log(p-1) - \frac{\log(p^2+1)}{2} - \tan^{-1}p\right] + c.$$

$$x = \frac{a}{2}\left[\log(p-1) - \tan^{-1}p\right] + c$$

$$\Rightarrow x = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{p^2+1}}\right. \longrightarrow (5)$$

$(1) \Rightarrow y = x + a\tan^{-1}p.$

$$\Rightarrow \frac{y-x}{a} = \tan^{-1}p.$$

$$\Rightarrow p = \tan\left(\frac{y-x}{a}\right)$$

(2) (ii) HW.

$y = x + 2\tan^{-1}p.$

Ans. $2x = \frac{(p-1)^2}{p^2+1}$, $y = x + 2\tan^{-1}p.$

③ Solve $3x + \log p - y = 0$.

Solution:

Given equation is

$3x + \log p - y = 0 \rightarrow ①$

Find y from ①.

$\Rightarrow y = 3x + \log p \rightarrow ②$

Diff. ② w.r.t. x we get

$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$.

But $\frac{dy}{dx} = p$.

$\Rightarrow p = 3 + \frac{1}{p}\frac{dp}{dx} \rightarrow ③$

which is I order, I degree D.E. with variables p & x.

Solving ③, $p - 3 = \frac{1}{p}\frac{dp}{dx}$.

$\Rightarrow dx = \frac{dp}{p(p-3)}$,

$\Rightarrow \int dx = \int \frac{dp}{p(p-3)} \longrightarrow ④$.

Now $\frac{1}{p(p-3)} = \frac{A}{p} + \frac{B}{p-3}$.

$\Rightarrow 1 = A(p-3) + Bp$.

$p = 0 \Rightarrow \boxed{A = -\frac{1}{3}}$ $p = 3 \Rightarrow \boxed{B = \frac{1}{3}}$

$\therefore ④ \Rightarrow x = \int \frac{(-1/3)}{p} dp + \int \frac{(\frac{1}{3})dp}{p-3}$.

$\Rightarrow x = -\frac{1}{3}\log p + \frac{1}{3}\log(p-3) + c$,

$\Rightarrow x = \frac{1}{3}\log \frac{p-3}{p} + c \rightarrow ④$.

Find value of p from ① or ④.

① $\Rightarrow \log p = y - 3x$.

$\Rightarrow \boxed{p = e^{y-3x}}$

Substituting in ④, we get general solution as

$$\boxed{x = \frac{1}{3}\log\left(\frac{e^{y-3x} - 3}{e^{y-3x}}\right) + c}$$

④ $y - 2px = \tan^{-1}(xp^2)$.

Solution:

Given: $y - 2px = \tan^{-1}(xp^2) \rightarrow ①$

Find y from ①.

$\Rightarrow y = 2px + \tan^{-1}(xp^2) \rightarrow ②$

Diff. ② w.r.t. x, we get.

$$\frac{dy}{dx} = 2\left[p.1 + x\frac{dp}{dx}\right] + \left[\frac{1}{1+(xp^2)^2}\right]\left[x.2p\frac{dp}{dx} + p^2.1\right]$$

$\frac{dy}{dx} = p \Rightarrow$

$$p = 2p + 2x\frac{dp}{dx} + \frac{2px\frac{dp}{dx} + p^2}{1+(xp^2)^2} \rightarrow ③$$

③ is I order I degree D.E. in variables p & x.

Solving ③, we get

$$-p - 2x\frac{dp}{dx} = p\frac{(p + 2x\frac{dp}{dx})}{1+(xp^2)^2}$$

$$\Rightarrow -\left[1 + (xp^2)^2\right]\left(p + 2x\frac{dp}{dx}\right) = p\left(p + 2x\frac{dp}{dx}\right).$$

$$\Rightarrow \left(p + 2x\frac{dp}{dx}\right)\left[p + \left(1+(xp^2)^2\right)\right] = 0$$

$$\Rightarrow p + 2x\frac{dp}{dx} = 0 \rightarrow (4)$$

$$\& \quad p + \left[1+(xp^2)^2\right] = 0 \rightarrow (5)$$

Solving (4), we get

$$-p = 2x\frac{dp}{dx}.$$

$$\Rightarrow 2\frac{dp}{p} = -\frac{dx}{x}$$

Integrating, we get

$$\log p^2 = -\log x + \log c$$

$$\Rightarrow \boxed{p^2 x = c} \rightarrow (A).$$

Find the value of p either from (1) or (A).

$$p^2 x = c \Rightarrow p^2 = \frac{c}{x},$$

$$\Rightarrow \boxed{p = \sqrt{c/x}}$$

Substituting in (1), we get general solution as

$$y - 2\sqrt{\frac{c}{x}}\cdot x = \tan^{-1}\left(x\cdot\frac{c}{x}\right)$$

$$\Rightarrow \boxed{y - 2\sqrt{cx} = \tan^{-1} c}.$$

(5) Solve $y + px = x^4 p^2$.

Solution:

Given: $y + px = x^4 p^2 \rightarrow (1)$

(1) is I order I degree D.E in variables p & x.

Diff. w.r.t x, we get

$$\frac{dy}{dx} + p\cdot 1 + x\frac{dp}{dx} = 4x^3p^2 + x^4\, 2p\cdot\frac{dp}{dx}.$$

But $\frac{dy}{dx} = p$.

$$\Rightarrow p + p + x\frac{dp}{dx} = 4x^3p^2 + 2x^4 p\frac{dp}{dx}.$$

$$\Rightarrow \frac{dp}{dx}\left(2px^4 - x\right) + 4x^3p^2 - 2p = 0.$$

$$\Rightarrow x\frac{dp}{dx}\left(2px^3 - 1\right) + 2p\left(2px^3 - 1\right) = 0$$

$$\Rightarrow x\frac{dp}{dx} + 2p = 0 \rightarrow (2)$$

$$\& \quad 2px^3 - 1 = 0 \rightarrow (3)$$

$$(2) \Rightarrow x\frac{dp}{dx} = -2p.$$

$$\Rightarrow \frac{dp}{p} = -2\frac{dx}{x}$$

Integrating, $\log p = -2\log x + \log c_1$

$$\Rightarrow \log px^2 = \log c.$$

$$\Rightarrow \boxed{px^2 = c} \Rightarrow p = \frac{c}{x^2}.$$

Sub. in (1), we get

$$y + \left(\frac{c}{x^2}\right)x = \left(\frac{c}{x^2}\right)^2 x^4.$$

$$\Rightarrow \boxed{xy + c = c^2x^2}.$$

(6) Solve $p^2+px^3-2x^2y=0$.

Solution:

$$p^2+px^3-2x^2y=0 \rightarrow (1)$$

Determine y.

$$p^2+px^3 = 2x^2y$$

$$\Rightarrow y = \frac{1}{2x^2}\left(p^2+px^3\right) \rightarrow (2)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = p = \frac{1}{2x^2}\left[2p\frac{dp}{dx}+3x^2p+x^3\frac{dp}{dx}\right]$$

$$+\left(p^2+px^3\right)\left(\frac{1}{2}(-2)x^{-3}\right).$$

$$\Rightarrow p = \frac{1}{2x^2}\left[(2p+x^3)\frac{dp}{dx}+3px^2\right]$$
$$-\frac{1}{x^3}\left(p^2+px^3\right).$$

$$\Rightarrow 2x^3p = x\left[(2p+x^3)\frac{dp}{dx}+3px^2\right]$$
$$-2\left(p^2+px^3\right).$$

$$\Rightarrow 2x^3p = (2p+x^3)x\frac{dp}{dx}+3px^3$$
$$-2p^2-2px^3.$$

$$\Rightarrow 2px^3-px^3+2p^2 = (2p+x^3)x\frac{dp}{dx}.$$

$$\Rightarrow 2p^2+px^3 = (2p+x^3)x\frac{dp}{dx}.$$

$$\Rightarrow p(2p+x^3) = (2p+x^3)x\frac{dp}{dx}.$$

$$\Rightarrow p = x\frac{dp}{dx}.$$

$$\Rightarrow \frac{dp}{p} = \frac{dx}{x}$$

Integrating, we get

$$\log p = \log x + \log c.$$

$$\Rightarrow \boxed{p = xc}$$

$\therefore (2) \Rightarrow$

$$y = \frac{1}{2x^2}\left[x^2c^2+x^4c\right].$$

$$= \frac{c^2}{2}+\frac{cx^2}{2}.$$

$$\Rightarrow \boxed{2y = c^2+cx^2}$$

(7) Solve $xp^2-yp-x=0$.

Solution:

$$xp^2-yp-x=0 \rightarrow (1)$$

Find y.

$$(1) \Rightarrow xp^2-x = yp.$$

$$\Rightarrow y = xp-\frac{x}{p} \rightarrow (2)$$

Diff. w.r.t. x, we get

$$\frac{dy}{dx} = p = p+x\frac{dp}{dx}-\frac{p-x\frac{dp}{dx}}{p^2}.$$

$$\Rightarrow 0 = x\frac{dp}{dx}-\frac{1}{p}+\frac{x}{p^2}\frac{dp}{dx}.$$

$$\Rightarrow \frac{1}{p} = x\left(1+\frac{1}{p^2}\right)\frac{dp}{dx}.$$

$$\Rightarrow \frac{dx}{x} = \frac{p^2+1}{p}dp.$$

Integrating, $\Rightarrow \log x = \frac{p^2}{2}+\log p+\log c.$

$$\Rightarrow \log\frac{x}{pc} = p^2/2.$$

$$\Rightarrow \boxed{x = cpe^{p^2/2}} \rightarrow (3)$$

$$(2) \& (3) \Rightarrow y = \frac{x(p^2-1)}{p} = cpe^{p^2/2}\frac{(p^2-1)}{p}$$

$$\Rightarrow y = ce^{p^2/2}(p^2-1) \rightarrow (4).$$

Elimination of p from (3) & (4), we get the general solution.

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(8) Solve $y = p\sin p + \cos p$.

Solution:

Given $y = p\sin p + \cos p \rightarrow (1)$.

Diff. (1) w.r.t x, we get

$$\frac{dy}{dx} = p = p\cos p\frac{dp}{dx} + \sin p\frac{dp}{dx} - \sin p\frac{dp}{dx}.$$

$$\Rightarrow p = p\cos p\frac{dp}{dx},$$

$$\Rightarrow 1 = \cos p\frac{dp}{dx}.$$

$$\Rightarrow dx = \cos p\, dp.$$

Integrating we get $x = \sin p + c \rightarrow (2)$

Eliminating p from (1) & (2) we get the general solution.

(9) Solve $xp^2 - 2yp + x = 0$.

Solution:

Given $xp^2 - 2yp + x = 0 \rightarrow (1)$

Find y.

$$(1) \Rightarrow xp^2 + x = 2yp.$$

$$\Rightarrow y = \frac{x(p^2+1)}{2p}.$$

Diff w.r.t. x, we get

$$\frac{dy}{dx} = p = \frac{p^2+1}{2p} + x\cdot\frac{p^2-1}{2p^2}\cdot\frac{dp}{dx}.$$

$$\frac{p^2-1}{2p} = \frac{dp}{dx}\cdot x\frac{(p^2-1)}{2p^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{dp}{p}.$$

Integrating $\boxed{p = cx}$

Eliminating p between $p = cx$ and (1), the solution is

$$\boxed{2cy = c^2x^2 + 1}$$

HW

(10) $y = 2px + y^2p^3$.

Ans: $y^2 = 2cx + c^3$

(11) $y = 2px + xp^2$.

Ans. $y = \pm 2c\sqrt{x} + c^2$

(12) $y = 2px + x^4$

Ans: $y = -8/7\, x^{9/2} + x^4$.

(13) $p^2 - xp + y = 0$

Ans. $y = cx - c^2$

(14) $xp^2 + 2px = y$

Ans. $(y-c)^2 = 4cx$.

Type III.
Equations solvable for x

Working Rule (Procedure)

Given D.E. is $f(x,y,p)=0 \longrightarrow (1)$

(1) is of I order and higher degree.

Determine x from (1), we get

$x=\phi(y,p) \longrightarrow (2)$

Differentiate (2) w.r.t. y we get

$$\frac{dx}{dy}=\varphi\left(y,p,\frac{dp}{dy}\right).$$

But $\frac{dy}{dx}=p$, $\frac{dx}{dy}=\frac{1}{p}$.

$\therefore \frac{1}{p}=\varphi\left(y,p,\frac{dp}{dy}\right) \longrightarrow (3)$

(3) is of I order I degree D.E. of variables y & p.

Solving (3), we get $F(y,p,c)=0 \longrightarrow (4)$

Find the value of p either from (1) or (4) and we get the general solution by substituting the value of p in the other equation.

Problems - (1) Solve $2px+y^2p^3-y=0$.

Solution: Given $2px+y^2p^3-y=0 \longrightarrow (1)$

Determine x from (1).

$\Rightarrow 2px=y-y^2p^3$

$\Rightarrow x=\frac{y-y^2p^3}{2p}$

$\Rightarrow x=\frac{y}{2p}-\frac{y^2p^2}{2}$

$\Rightarrow x=\frac{1}{2}yp^{-1}-\frac{1}{2}y^2p^2 \longrightarrow (2)$

Diff. (2) w.r.t. y, we get

$$\frac{dx}{dy}=\frac{1}{2}\left[\left(1\cdot p^{-1}+y(-1)p^{-2}\frac{dp}{dy}\right)-\left(2yp^2+y^2\cdot 2p\frac{dp}{dy}\right)\right].$$

But $\frac{dx}{dy}=1/p$.

$\Rightarrow \frac{1}{p}=\frac{1}{2}\left[\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}\right]-\frac{1}{2}\left[2y^2p\frac{dp}{dy}+2yp^2\right].$

$\Rightarrow \frac{1}{p}=\frac{1}{2p}-\frac{y}{2p^2}\frac{dp}{dy}-y^2p\frac{dp}{dy}-yp^2.$

$\Rightarrow \frac{1}{p}-\frac{1}{2p}+yp^2=-y\frac{dp}{dy}\left(\frac{1}{2p^2}+yp\right).$

$\Rightarrow \frac{2-1+2yp^3}{2p}=-y\frac{dp}{dy}\left(\frac{1+2yp^3}{2p^2}\right).$

$\Rightarrow \frac{1+2yp^3}{2p}=-\frac{1+2yp^3}{2p^2}\cdot y\frac{dp}{dy}.$

$\Rightarrow 1=-\frac{1}{p}y\frac{dp}{dy} \longrightarrow (3)$

(3) is I order I degree D.E.

Using variable separable form, we have $\frac{dy}{y}=-\frac{dp}{p}$.

Integrating $\log y=-\log p+\log c$

$\Rightarrow yp=c \longrightarrow (4)$

Determine p from (1) or (4)

(4) $\Rightarrow p=c/y$.

Substituting in (1), we get

$2px+y^2p^3-y=0.$

$\Rightarrow 2\cdot\frac{c}{y}x+y^2\frac{c^3}{y^3}-y=0.$

$\Rightarrow \frac{2cx}{y}+\frac{c^3}{y}-y=0.$

$\Rightarrow \frac{2cx+c^3-y^2}{y}=0$

$\Rightarrow \boxed{2cx+c^3-y^2=0}$

Problem ②

Solve $p^2 - 4xyp + 8y^2 = 0$.

Solution: Given $p^3 - 4xyp + 8y^2 = 0 \rightarrow ①$

Determine x from ①

$\Rightarrow -4xyp = -p^3 - 8y^2$

$\Rightarrow x = \frac{p^3 + 8y^2}{4yp}$

$\Rightarrow x = \frac{p^3}{4yp} + \frac{8y^2}{4yp}$

$\Rightarrow x = \frac{p^2}{4y} + \frac{2y}{p} \rightarrow (*)$

$\Rightarrow x = \frac{1}{4} p^2 y^{-1} + 2yp^{-1} \rightarrow ②$

Diff. ② w.r.t. y, we get

$\frac{dx}{dy} = \frac{1}{4}\left[p^2(-1)y^{-2} + y^{-1} \cdot 2p\frac{dp}{dy}\right] + 2\left[y(-1)p^{-2}\frac{dp}{dy} + 1 \cdot p^{-1}\right]$.

But $\frac{dx}{dy} = \frac{1}{p}$.

$\Rightarrow \frac{1}{p} = \frac{-p^3}{4y^2} + \frac{p}{2y}\frac{dp}{dy} - \frac{2y}{p^2}\frac{dp}{dy} + \frac{2}{p}$

$\Rightarrow \frac{1}{p} - \frac{2}{p} + \frac{p^2}{4y^2} = \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right)$.

$\Rightarrow \frac{4y^2 - 8y^2 + p^3}{4y^2 p} = \frac{dp}{dy}\left(\frac{p^3 - 4y^2}{2p^2 y}\right)$.

$\Rightarrow \frac{p^3 - 4y^2}{4y^2 p} = \frac{p^3 - 4y^2}{2p^2 y} \cdot \frac{dp}{dy}$.

$\Rightarrow \frac{1}{2y} = \frac{1}{p}\frac{dp}{dy} \rightarrow ③$

③ is I order I degree D.E. using variable separable method, we get $\frac{dy}{2y} = \frac{dp}{p}$.

Integrating, $\frac{1}{2}\log y = \log p + \log c$

$\Rightarrow \log \frac{y^{+1/2}}{p} = \log c$.

$\Rightarrow \frac{y^{1/2}}{p} = c$

$\Rightarrow p = \frac{\sqrt{y}}{c}$.

Substituting in ①, we get

$\left(\frac{\sqrt{y}}{c}\right)^3 - 4xy(\sqrt{y}/c) + 8y^2 = 0$.

$\Rightarrow \frac{y\sqrt{y}}{c^3} - \frac{4xy\sqrt{y}}{c} + 8y^2 = 0$.

$\sqrt{y} - 4xc^2\sqrt{y} + 8c^3 y = 0$.

$\Rightarrow \sqrt{y} - 4xc^2 + 8c^3\sqrt{y} = 0$.

$\Rightarrow 1 - 4xc^2 + 8c^3\sqrt{y}$

which is the general solution.

Problem ③ $y = 3px + 6p^2y^2$.

Solution: Given $y = 3px + 6p^2y^2 \rightarrow ①$

Find x from ①

$\Rightarrow 3x = \frac{y - 6p^2y^2}{p}$.

Diff. w.r.t. y we get

$3\frac{dx}{dy} = \frac{p\left[1 - 12p^2 y - 12y^2 p\frac{dp}{dy}\right]}{p^2} - \frac{(y - 6p^2y^2)\frac{dp}{dy}}{p^2}$

$\Rightarrow \frac{3}{p} = \frac{p - 12yp^3 - 12y^2p^2\frac{dp}{dy} - y\frac{dp}{dy} + 6y^2p^2\frac{dp}{dy}}{p^2}$

$\Rightarrow 3p = p - 12yp^3 + \frac{dp}{dy}\left(6y^2p^2 - y - 12y^2p^2\right)$.

$\Rightarrow 2p + 12yp^3 = -(y + 6y^2p^2)\frac{dp}{dy}$.

$\Rightarrow 2p(1 + 6yp^2) + (1 + 6yp^2)y\frac{dp}{dy} = 0$.

$\Rightarrow \left(2p + y\frac{dp}{dy}\right)(1 + 6yp^2) = 0$.

$\Rightarrow 2p + y\frac{dp}{dy} = 0 \rightarrow ②$

& $1 + 6yp^2 = 0 \rightarrow ③$

Now ② ⇒

$2p + y\frac{dp}{dy} = 0.$

$\Rightarrow 2p = -y\frac{dp}{dy}$

$\Rightarrow 2\frac{dy}{y} = -\frac{dp}{p}.$

Integrating, we get

$2\log y = -\log p + \log c$

$\Rightarrow \log y^2 + \log p = \log c$

$\Rightarrow y^2 p = c \Rightarrow \boxed{p = \frac{c}{y^2}}$

Sub. in ①, we get

$y = 3px + 6p^2y^2$

$\Rightarrow y = 3\left(\frac{c}{y^2}\right)x + 6\left(\frac{c}{y^2}\right)^2 y^2$

$\Rightarrow y = \frac{3cx}{y^2} + \frac{6c^2}{y^2}$

$\Rightarrow \boxed{y^3 = 3cx + 6c^2}$

④ Solve $yp^2 - xp + 2y = 0.$

Solution: Given $yp^2 - xp + 2y = 0 \rightarrow ①$

Solving for x, we get

$x = yp + \frac{2y}{p} \rightarrow ②$

Diff. w.r.t. y, we get

$\frac{dx}{dy} = \frac{1}{p} = y\frac{dp}{dy} + p + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$

$\Rightarrow \frac{1}{p} = p + \frac{2}{p} + \left(y - \frac{2y}{p^2}\right)\frac{dp}{dy}.$

$\Rightarrow \frac{1}{p} - p - \frac{2}{p} = \frac{p^2-2}{p^2}\cdot y\frac{dp}{dy}.$

$\Rightarrow \frac{p^2+2-1}{p} + \frac{p^2-2}{p^2}y\frac{dp}{dy} = 0.$

$\Rightarrow \frac{p^2+1}{p} + \frac{p^2-2}{p^2}y\frac{dp}{dy} = 0.$

$\Rightarrow (p^2+1) + \frac{p^2-2}{p}y\frac{dp}{dy} = 0$

$\Rightarrow \frac{p^2-2}{p}y\frac{dp}{dy} = -(p^2+1).$

$\Rightarrow \frac{p^2-2}{p(p^2+1)}dp = -\frac{dy}{y}.$

$\Rightarrow \frac{dy}{y} + \frac{p^2-2}{p(p^2+1)}dp = 0 \rightarrow ③$

Now $\frac{p^2-2}{p(1+p^2)} = \frac{A}{p} + \frac{Bp+C}{1+p^2}.$

$\Rightarrow p^2 - 2 = A(1+p^2) + (Bp+C)p.$

Coeff. of $p^2 \Rightarrow 1 = A + B$

Constant $\Rightarrow \boxed{-2 = A}$

$A + B = 1 \Rightarrow -2 + B = 1 \Rightarrow \boxed{B = 3}$

Coeff of $p \Rightarrow \boxed{0 = C}$

$\therefore ③ \Rightarrow \frac{dy}{y} + \left[\frac{-2}{p} + \frac{3p+0}{p^2+1}\right]dp = 0.$

$\Rightarrow \frac{dy}{y} + \left(\frac{3p}{1+p^2} - \frac{2}{p}\right)dp = 0.$

Integrating, we get

$\log y + \frac{3}{2}\log(1+p^2) - 2\log p = \log c$

$\Rightarrow \log y + \log(1+p^2)^{3/2} - \log p^2 = \log c$

$\Rightarrow \log y = \log c + \log p^2 - \log(1+p^2)^{3/2}.$

$\Rightarrow y = \frac{cp^2}{(1+p^2)^{3/2}} \rightarrow ④$

$① \Rightarrow x = \frac{y}{p}(p^2+2)$

$\Rightarrow x = \frac{cp^2}{p(1+p^2)^{3/2}}(p^2+2)$

$\Rightarrow x = \frac{cp(p^2+2)}{(1+p^2)^{3/2}} \rightarrow ⑤$

Eliminating p from ④ & ⑤, we get the general solution.

Problems-

(5) (i) $x = yp + p^2$ (ii) $x - yp = ap^2$.

(6) $p^3y + 2px = y$.

(7) $yp^2 - 2xyp + y = 0$

(8) $p^2 + xpy^2 + y^3 = 0$.

(9) $xp^2 + yp = 3y^4$.

(10) $9y^2p^2 - 3xp + y = 0$.

(11) $x^2 = 1 + p^2$.

(12) $x(1+p^2) = 1$.

(13) $x = y + a \log p$.

(14) $y^2 \log y = xyp + p^2$.

(15) $x - \tan^{-1} p - \frac{p}{1+p^2} = 0$.

Answers- (5) i) $y = -ap + \frac{1}{\sqrt{1-p^2}}(c + \sin^{-1} p)$

$$x = \frac{p}{\sqrt{1-p^2}}(c + \sin^{-1} p)$$

(7) $y = \frac{p^2 - 2\log p}{4}$, $x = \frac{1+p^2}{p}$.

(8) $cy(x-c) = 1$.

(9) $3y = c(1 + cxy)$

(10) $y^3 = c(x - c)$

(11) $x = \sqrt{1+p^2}\, y$, $y = \frac{p}{2}\sqrt{1+p^2} - \frac{1}{2}\sinh^{-1} p$

(12) $y = \frac{-2p}{1+p^2}$, $x = \frac{1}{1+p^2}$.

(16) Solve $x = y^2 + \log p$.

Solution: Given $x = y^2 + \log p \rightarrow (1)$

This is solvable for x.

Diff w.r.t. y

$$\frac{dx}{dy} = \frac{1}{p} = 2y + \frac{1}{p}\frac{dp}{dy}.$$

$$\Rightarrow \frac{dp}{dy} + 2yp = 1.$$

This is linear in p.

$$IF = e^{\int 2y\,dy} = e^{y^2/2}$$

$\therefore$ Solution is

$$p e^{y^2/2} = \int e^{y^2} dy + c \rightarrow (2)$$

Eliminating p between (1) & (2) gives the general solution.

Type IV Clairaut's Equation

An equation of the form $y = px + f(p)$ $\rightarrow$ (1) is known as Clairaut's equation.

Differentiating w.r.t. x, we have

$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$.

$\Rightarrow \left(x + f'(p)\right)\frac{dp}{dx} = 0$.

$\Rightarrow \frac{dp}{dx} = 0$ (or) $x + f'(p) = 0$

$\frac{dp}{dx} = 0 \Rightarrow p = c \rightarrow$ (2).

Eliminating p from (1) & (2),

we get $y = cx + f(c) \rightarrow$ (3)

which is the general solution of (1).

$\therefore$ The solution of the Clairaut's equation is obtained on replacing p by c.

To find the singular solution, we have the following steps:

(i) Find the general solution by replacing p by c. (i.e (3))

(ii) Diff. this w.r.t $c \Rightarrow x + f'(c) = 0$ $\rightarrow$ (4)

(iii) Eliminate c from (3) & (4), we get the singular solution.

Problem (1) Solve $y = (x-a)p - p^2$.

Solution: Given $y = (x-a)p - p^2 \rightarrow$ (1)

This is of Clairaut's form

$y = px + f(p)$.

The solution is got by putting $p = c$.

$\Rightarrow y = cx + f(c)$.

(i.e) $\boxed{y = (x-a)c - c^2}$

Problem (2) Solve $y = 2px + y^2p^2$.

Solution: Given $y = 2px + y^2p^2 \rightarrow$ (1)

Put $X = 2x$, $Y = y^2$

$\Rightarrow dX = 2dx$, $dY = 2y\,dy$.

$\therefore P = \frac{dY}{dX} = \frac{2y\,dy}{2dx} = yp$. $\left(\because \frac{dy}{dx} = p\right)$

$\therefore$ The equation transforms into $Y = XP + P^2$

This is Clairaut's equation

$\Rightarrow Y = cX + c^2$. [$\because P = c$]

$\Rightarrow$ Solution is

$\boxed{y^2 = 2xc + c^2}$

Problem (3)

Solve $(px - y)(py + x) = a^2p$ $\rightarrow$ (1)

Solution:

Put $x^2 = u$ & $y^2 = v$.

$\Rightarrow 2x\,dx = du$ & $2y\,dy = dv$

$\therefore P = \frac{dv}{du} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}\frac{dy}{dx} = \frac{y}{x}p$.

$\Rightarrow p = \frac{x}{y}P$.

$\therefore$ (1) $\Rightarrow (uP - v)(P+1) = a^2P$.

$\Rightarrow uP - v = \frac{a^2P}{P+1}$

$\Rightarrow v = uP - \frac{a^2P}{P+1}$ which is Clairaut's form.

$\therefore$ Its solution is

$v = uc - \frac{a^2c}{c+1} \Rightarrow y^2 = cx^2 - \frac{a^2c}{c+1}$.

(4) $(y-px)(p-1)=p$. vittal

(5) $y = xp + \frac{a}{p}$.

(6) $(px-y)(x+py) = 2p$.

[Ans. $y^2 = cx^2 - \frac{2c}{c+1}$].